Rotationally invariant order parameter equations for natural patterns in nonequilibrium systems

M. Bestehorn and R. Friedrich
Institut für Theoretische Physik und Synergetik, Universität Stuttgart, Pfaffenwaldring 57/4, 70550 Stuttgart, Germany

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We discuss a theoretical description of the formation of cellular patterns exhibiting defects, grain boundaries, and spiral patterns in nonequilibrium large-aspect-ratio systems by means of rotationally invariant order parameter equations. Starting from evolution equations of general form, we show that the order parameter equations which can be derived close to a bifurcation point in general involve nonlinear terms which are nonlocal. We present a suitable approximation scheme of these terms by local ones which is based on a gradient expansion. A truncation of this expansion leads to model equations which are widely used in the theoretical treatment of natural patterns in complex systems. [S1063-651X(99)01003-X]

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I. INTRODUCTION

There are numerous examples of systems far from equilibrium which exhibit an instability leading to the formation of well-ordered patterns [1–5]. In spatially extended systems various planforms are observed like roll structures, square patterns, and hexagonal patterns. Well-known examples include hydrodynamic systems, like Rayleigh-Bénard convection with its various modifications [6–8], the Faraday instability [9], chemical instabilities leading to Turing structures [10–12], and pattern formation in the transverse field of lasers [13].

All above mentioned systems share the common property that they have, for a certain range of control parameters, a stable stationary state which is, for large-aspect-ratio systems, independent of the horizontal coordinates $\mathbf{x}=(x,y)$. As one or several control parameters are changed this state undergoes an instability, leading to the formation of cellular patterns.

The theoretical treatment of these pattern forming systems starts with an investigation of the basic equations describing the various systems under considerations. These equations are the well-known hydrodynamic equations, laser equations, or reaction-diffusion equations. They take the general form

$$\dot{\mathbf{q}}(\mathbf{r},t)=\mathbf{N}(\mathbf{q}(\mathbf{r},t),\boldsymbol{\nabla},\sigma), \tag{1.1}$$

where $\mathbf{N}$ is a nonlinear function of the state vector $\mathbf{q}(\mathbf{r},t)$ as well as its spatial derivatives. Furthermore it depends on a set of control parameters σ. Since similar patterns are observed in quite different systems, a unified mathematical description should be possible. As is now well established [14,3] such a reduced description becomes possible close to instability. The state vectors depend on order parameters for which a closed system of evolution equations can be obtained.

Three different kinds of reduced descriptions have been studied in the past, each focusing on certain aspects of the pattern forming system. The first approach is devoted to perfect patterns which are defined on a spatially periodic lattice. The order parameters are amplitudes of modes with a fixed spatial periodicity. Let us consider the simplest case of a perfect roll structure. Here the state vector is described as

$$\mathbf{q}(\mathbf{r},t)=\xi(t)\boldsymbol{\Phi}(z)\exp(ik_c x)+\text{c.c.}+O(\xi^2), \tag{1.2}$$

where the order parameter obeys the Landau amplitude equation

$$\dot{\xi}(t)=\varepsilon\,\xi(t)-a|\xi(t)|^2\xi(t). \tag{1.3}$$

Hexagonal structures are defined on a hexagonal lattice, and involve three complex order parameters, etc. More complicated situations are investigated using group theoretic methods [15,16].

A second kind of reduced description deals with almost perfect patterns. Slow spatial variations of the order parameters are taken into account. For the case of roll structures the state vector takes the form

$$\mathbf{q}(\mathbf{r},t)=\xi(\mathbf{x},t)\boldsymbol{\Phi}(z)\exp(ik_c x)+\text{c.c.}+O(\xi^2), \tag{1.4}$$

and the order parameter obeys a partial differential equation of the form of a Ginzburg-Landau equation [17–19]:

$$\dot{\xi}(\mathbf{x},t)=\left[\varepsilon+\left(\partial_x+\frac{i}{\sqrt{2}}\partial_y^2\right)^2\right]\xi(\mathbf{x},t)-a|\xi(\mathbf{x},t)|^2\xi(\mathbf{x},t). \tag{1.5}$$

Extensions to square and hexagonal patterns are obvious [20,21]. The equations for slowly varying amplitudes are able to describe spatially slow variations of cellular structures, including defects of the regular lattice [22]. However, the following physical effects are missed. First, the envelope equations lack a rotational symmetry, and are therefore not able to cover, for instance, target patterns or spiral patterns. Second, the amplitude equations are invariant with respect to phase shifts of the form

$$\xi(\mathbf{x},t)\rightarrow e^{i\alpha}\xi(\mathbf{x},t). \tag{1.6}$$

As can be seen from Eq. (1.4), such a phase shift belongs to a mere translation of the envelope structure described by ξ, i.e., of defects or grain bondaries of a roll pattern. No interaction between the defect and the underlying roll structure occurs. Therefore, the phenomenon of pinning of defects to the underlying roll structure is not included. Pinning is a nonperturbative effect which is missing in the reduced description using envelope equations [23]. Third, if we consider stationary solutions of the one-dimensional Ginzburg-Landau equation, one may show that there are only spatially periodic or quasiperiodic solutions. However, there are general arguments which indicate that there also exist spatially chaotic stationary solutions of the basic equations [24]. These chaotic solutions are missed in the description of the pattern forming process using spatially slowly varying amplitudes.

The third kind of reduced description is based on an order parameter which obeys a rotationally invariant evolution equation containing the spatially fast modulations of the emerging cellular structures [14,25]. The order parameter field can be directly connected with the structures seen in the two-dimensional plane. Such an equation can be obtained from the basic equations, provided the behavior in the vertical (z) direction is enslaved by the spatiotemporal structures in the horizontal plane. It is evident that the above mentioned reduced descriptions can be derived further from this order parameter equation.

Since the order parameter equation is rotationally invariant and incorporates fast spatial variations, it contains complicated nonlinear terms which are decisive for a description of the selection of the various planforms as well as the instabilities of the perfect cellular patterns. Therefore, the question arises of whether these terms may be further simplified. As is well known, a substitution of these terms by simpler ones leads to model equations like the so-called Swift-Hohenberg equation [25], which may account for some aspects of pattern formation but may fail in reproducing, for instance, the correct stability boundaries for rolls. Usually, the stability boundaries of the roll solutions with respect to large wavelength modulations of the zigzag instability, as well as the cross roll instability, depend decisively on the structure of the nonlinear interaction terms. Therefore, a mere substitution of these terms, e.g., by a simple cubic term, is not justified.

The present paper is devoted to a discussion of the interaction terms which are in general nonlocal, and their approximation by terms involving the order parameter field as well as their spatial derivatives. The paper is outlined as follows. In Sec. II we summarize the derivation of the order parameter equation. Then we consider a representation of the nonlinear terms invoking symmetry arguments leading to the expansion of the nonlocal interaction terms by local ones. We consider the case of an instability involving one order parameter for systems both with and without reflectional symmetry. Furthermore, the case of an instability involving order parameters of pseudoscalar type is considered. In the appendix we consider the long wavelength instabilities as well as the amplitude instability for roll solutions of the order parameter equation for systems with reflectional symmetry. We discuss how the corresponding stability boundaries are affected by the truncation of the expansion of the nonlinear terms.

II. ORDER PARAMETER EQUATIONS

We shall investigate evolution laws of the form

$$\dot{\mathbf{q}}(\mathbf{r},t)=L(\boldsymbol{\nabla},\boldsymbol{\sigma})\mathbf{q}(\mathbf{r},t)+\Gamma:\mathbf{q}(\mathbf{r},t):\mathbf{q}(\mathbf{r},t). \tag{2.1}$$

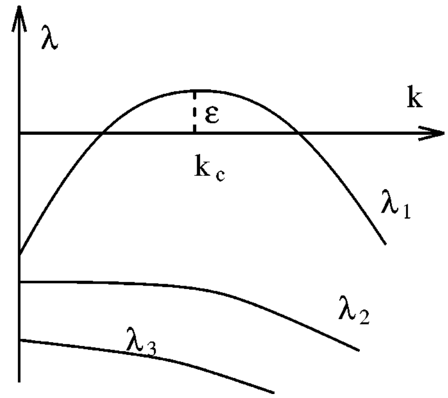

FIG. 1. The systems under consideration possess eigenvalues that are grouped in several (usually infinitely many) bands. The above band may cross the k axis for certain values of the control parameter, and mark the onset of an instability with a typical wave number k_c. Modes that belong to this band are called order parameters. All other bands stay below the k axis for all parameter values. They belong to linearly damped modes which can be eliminated.

Thus we restrict our attention to systems with a quadratic nonlinearity. The state vector $\mathbf{q}(\mathbf{r},t)$ describes the deviation from a basic state, which is time independent and homogeneous in the horizontal directions (x,y).

The stability of the basic state $\mathbf{q}=\mathbf{0}$ is investigated by a normal mode analysis, neglecting the nonlinearities. Due to translational symmetry in the horizontal plane the normal modes take the following form $[\mathbf{x}=(x,y)]$:

$$\boldsymbol{\Phi}_{j,\mathbf{k}}(\mathbf{x},z)=\boldsymbol{\Phi}_j(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.2}$$

The corresponding eigenvalue problem reads

$$\lambda_j(\mathbf{k})\boldsymbol{\Phi}_j(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{x}}=L(\boldsymbol{\nabla},\boldsymbol{\sigma})\boldsymbol{\Phi}_j(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.3}$$

The discrete index j specifies the mode structure in vertical direction (see Fig. 1). The continuous wave vector $\mathbf{k}$ defines the orientation as well as wavelength of the plane waves.

In order to deal with the nonlinear properties the state vector $\mathbf{q}(\mathbf{r},t)$ is expanded into a complete set of normal modes:

$$\mathbf{q}(\mathbf{r},t)=\sum_j \int d\mathbf{k}\, \xi_j(\mathbf{k},t)\boldsymbol{\Phi}_j(\mathbf{k},z)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.4}$$

Inserting this expansion into the evolution equation, one can derive a set of differential equations for the mode amplitudes $\xi_j(\mathbf{k},t)$:

$$\begin{aligned}\dot{\xi}_j(\mathbf{k},t)&=\lambda_j(\mathbf{k})\xi_j(\mathbf{k},t)\\&\quad+\sum_{j',j''}\int d^2\mathbf{k}'\int d^2\mathbf{k}''\,\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\\&\quad\times\Gamma_{j;j',j''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')\xi_{j'}(\mathbf{k}',t)\xi_{j''}(\mathbf{k}'',t).\end{aligned} \tag{2.5}$$

Here the matrix elements $\Gamma_{j;j',j''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')$ are introduced according to:

$$\Gamma_{j;j',j''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')=\langle\boldsymbol{\Phi}^{\dagger}_{j,\mathbf{k}}(\mathbf{x},z)|\Gamma:\boldsymbol{\Phi}_{j',\mathbf{k}'}(\mathbf{x},z):\boldsymbol{\Phi}_{j'',\mathbf{k}''}(\mathbf{x},z)\rangle. \tag{2.6}$$

The brackets denote a suitably defined scalar product, and $\Phi^{\dagger}$ is the solution of the adjoint problem (2.3). Due to Eq. (2.3), the linear part of Eq. (2.5) is diagonal. The eigenvalues $\lambda_j(\mathbf{k})$ are continuous functions of the wave vector $\mathbf{k}$. For rotationally invariant systems they only depend on the modulus $|\mathbf{k}|$. There are $j=1,\ldots,\infty$ different bands of modes (Fig. 1). If the basic state is stable, all bands have negative growth rates $\mathrm{Re}[\lambda_j(\mathbf{k})]<0$. An instability arises if one (or several) bands have modes with vanishing or even positive growth rates, at least in a region around a certain critical wave vector k_c. The bands formed of modes with negative growth rates are denoted as stable bands.

We denote the mode amplitudes of the critical and the stable bands with $\xi_u(\mathbf{k},t)$, and $\xi_s(\mathbf{k},t)$ respectively. The amplitudes $\xi_u(\mathbf{k},t)$ define the order parameters. Since the growth rates of the stable modes all are negative, we can integrate the evolution equation for the modes of the stable bands to obtain

$$\xi_s(\mathbf{k},t)=\int_{-\infty}^{t} d\tau\, e^{\lambda_s(\mathbf{k})(t-\tau)} \sum_{u',u''} \int d^2\mathbf{k}' \int d^2\mathbf{k}''\, \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\Gamma_{s;u',u''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')\xi_{u'}(\mathbf{k}',\tau)\xi_{u''}(\mathbf{k}'',\tau). \tag{2.7}$$

Here only terms quadratic in the order parameters $\xi_u(\mathbf{k},t)$ are retained. Higher order terms can be determined iteratively.

An adiabatic elimination of the modes [1,14] of the stable bands leads to a closed set of evolution equations for the order parameters. In lowest order one obtains

$$\xi_s(\mathbf{k},t)=\sum_{u',u''} \int d^2\mathbf{k}' \int d^2\mathbf{k}''\, \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\frac{\Gamma_{s;u',u''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')}{i(\omega_{u'}(\mathbf{k}')+\omega_{u''}(\mathbf{k}''))-\lambda_s(\mathbf{k})}\xi_{u'}(\mathbf{k}',t)\xi_{u''}(\mathbf{k}'',t). \tag{2.8}$$

Here $\omega_u(\mathbf{k})$ denotes the imaginary part of the eigenvalue $\lambda_u(\mathbf{k})$.

Inserting the resulting expression for the stable modes into the equations for the unstable modes yields the following order parameter equations. Here we have only retained terms up to cubic orders:

$$\begin{aligned}\dot{\xi}_u(\mathbf{k},t)={}&\lambda_u(\mathbf{k})\xi_u(\mathbf{k},t)+\sum_{u',u''} \int d^2\mathbf{k}' \int d^2\mathbf{k}''\, \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\Gamma^2_{u;u',u''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')\xi_{u'}(\mathbf{k}',t)\xi_{u''}(\mathbf{k}'',t)\\ &+\sum_{u',u'',u'''} \int d^2\mathbf{k}' \int d^2\mathbf{k}'' \int d^2\mathbf{k}'''\, \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}''-\mathbf{k}''')\Gamma^3_{u;u',u'',u'''}(\mathbf{k};\mathbf{k}',\mathbf{k}'',\mathbf{k}''')\xi_{u'}(\mathbf{k}',t)\xi_{u''}(\mathbf{k}'',t)\xi_{u'''}(\mathbf{k}''',t).\end{aligned} \tag{2.9}$$

The mode coupling coefficients $\Gamma^3_{u;u',u'',u'''}(\mathbf{k};\mathbf{k}',\mathbf{k}'',\mathbf{k}''')$ take the form

$$\begin{aligned}\Gamma^3_{u;u',u'',u'''}(\mathbf{k};\mathbf{k}',\mathbf{k}'',\mathbf{k}''')={}&\sum_s \int d\mathbf{k}_s\, \delta(\mathbf{k}_s-\mathbf{k}'-\mathbf{k}'')(\Gamma_{u;u',s}(\mathbf{k};\mathbf{k}',\mathbf{k}_s)+\Gamma_{u;s,u'}(\mathbf{k};\mathbf{k}_s,\mathbf{k}'))\\ &\times\frac{\Gamma_{s,u'',u'''}(\mathbf{k}_s;\mathbf{k}'',\mathbf{k}''')}{i(\omega_{u''}(\mathbf{k}'')+\omega_{u'''}(\mathbf{k}'''))-\lambda_s(\mathbf{k}_s)}.\end{aligned} \tag{2.10}$$

It is convenient to transform the order parameter equation from $\mathbf{k}$ space into real space. We define order parameter fields $\Psi_u(\mathbf{x},t)$ by the Fourier transforms of the amplitudes $\xi_u(\mathbf{k},t)$:

$$\Psi_u(\mathbf{x},t)=\int d\mathbf{k}\, \xi_u(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.11}$$

These fields then obey the equation

$$\begin{aligned}\dot{\Psi}_u(\mathbf{x},t)={}&\lambda_u(-i\nabla)\Psi_u(\mathbf{x},t)+\sum_{u',u''} \int d^2\mathbf{x}'\, d^2\mathbf{x}''\, \Gamma^2_{u;u',u''}(\mathbf{x}-\mathbf{x}',\mathbf{x}-\mathbf{x}'')\Psi_{u'}(\mathbf{x}',t)\Psi_{u''}(\mathbf{x}'',t)\\ &+\sum_{u',u'',u'''} \int d^2\mathbf{x}'\, d^2\mathbf{x}''\, d^2\mathbf{x}'''\, \Gamma^3_{u;u',u'',u'''}(\mathbf{x}-\mathbf{x}',\mathbf{x}-\mathbf{x}'',\mathbf{x}-\mathbf{x}''')\Psi_{u'}(\mathbf{x}',t)\Psi_{u''}(\mathbf{x}'',t)\Psi_{u'''}(\mathbf{x}''',t).\end{aligned} \tag{2.12}$$

The kernels $\Gamma^2_{u;u',u''}(\mathbf{x}-\mathbf{x}',\mathbf{x}-\mathbf{x}'')$ are obtained from the Fourier transforms of the mode coupling coefficients $\Gamma^2_{u;u',u''}(\mathbf{k};\mathbf{k}',\mathbf{k}'')$, etc. It is evident how higher order terms have to be included.

We may summarize the preceding results as follows. The instability of a stationary spatially homogeneous state of a large-aspect-ratio system can be described by the order parameter fields $\Psi_u(\mathbf{x},t)$ which define the state vector according to:

$$\mathbf{q}(\mathbf{r},t)=\sum_u \boldsymbol{\Phi}_u(-i\nabla,z)\Psi_u(\mathbf{x},t)$$
$$+\sum_s \boldsymbol{\Phi}_s(-i\nabla,z)\Psi_s[\Psi_u(\mathbf{x},t)], \qquad (2.13)$$

The amplitudes of the stable modes are functions of the order parameter fields. These fields obey evolution equations of the form

$$\dot{\Psi}_u(\mathbf{x},t)=\lambda_u(-i\nabla)\Psi_u(\mathbf{x},t)+H_u[\Psi_u(\mathbf{x},t)], \qquad (2.14)$$

where $H_u[\Psi_u(\mathbf{x},t)]$ is a nonlinear nonlocal function. For the case of supercritical instabilities an approximation including cubic terms as in Eq. (2.12) is usually sufficient.

III. APPROXIMATION OF THE MODE COUPLING COEFFICIENTS

It is desirable to obtain suitable approximations for the linear operators $\lambda_u(-i\nabla)$ as well as for the kernels $\Gamma^2_{u;u',u''}(\mathbf{x}-\mathbf{x}',\mathbf{x}-\mathbf{x}'')$, etc., of the nonlinear terms. Let us start with the linear terms. Due to rotational symmetry in the horizontal plane, the eigenvalues in Eq. (2.3) only depend on k^2. If there is one band of unstable modes with real eigenvalues, $\lambda_u(k^2)$ can be expanded at k_c with respect to $k^2-k_c^2$:

$$\lambda_u(k^2)=\left.\frac{\partial\lambda_u}{\partial\sigma}\right|_{k_c\sigma_c}\sigma_c\varepsilon+\frac{1}{2}\left.\frac{\partial^2\lambda_u}{\partial(k^2)^2}\right|_{k_c\sigma_c}(k^2-k_c^2)^2. \qquad (3.1)$$

Furthermore, we have introduced a reduced control parameter $\varepsilon=\sigma/\sigma_c-1$ which measures the deviation from threshold.

Transforming expression (3.1) to real space leads, after an appropriate scaling of time and space, to the linear operator of the Swift-Hohenberg equation [25]:

$$\varepsilon-(1+\Delta)^2. \qquad (3.2)$$

Here and for the following, Δ denotes the Laplacian with respect to the horizontal coordinates $\mathbf{x}$. The extension to a pair of complex eigenvalues crossing the real axis demands for a complex order parameter field and includes a dispersion γ [26]:

$$\varepsilon+i\omega_u(k_c)-i\gamma(1+\Delta)-(1+\Delta)^2, \qquad (3.3)$$

with

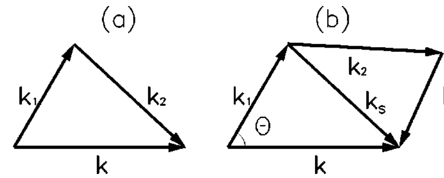

FIG. 2. Wave vector selection in Fourier space for quadratic (a) and cubic (b) nonlinear terms. (a) An arbitrary triangle is fixed by its three sides, leading to expressions in real space that include three Laplace operators like Eq. (3.28). (b) An arbitrary square needs for a unique description its four sides and one diagonal, for instance $\mathbf{k}_s=\mathbf{k}_2+\mathbf{k}_3$. This leads to cubic expressions including five Laplacians like Eq. (3.25).

$$\gamma\propto\left.\frac{\partial\omega_u}{\partial k^2}\right|_{k_c\sigma_c}.$$

An approximation of the nonlinear mode coupling coefficients turns out to be more involved. An important step consists of investigating the dependence of the mode coupling coefficients Γ^2 and Γ^3 in k space on the various $\mathbf{k}$ vectors. This dependence is to some extent determined by the underlying symmetries of the system.

For the sake of simplicity, from now on we assume the case of one order parameter field $\xi(\mathbf{k},t)$ or $\Psi(\mathbf{x},t)$; we therefore may drop the indices u and denote $\mathbf{k}'$ by $\mathbf{k}_1$, etc. The extension to more order parameter fields is evident.

Translational symmetry is responsible for the selection rules of the $\mathbf{k}$ vectors:

$$\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2 \quad \text{for} \quad \Gamma^2,$$
$$\mathbf{k}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 \quad \text{for} \quad \Gamma^3. \qquad (3.4)$$

Rotational symmetry implies that the mode coupling coefficients depend only on the absolute values of the wave vectors, the scalar products, and the vertical components of the cross products between the various wave vectors involved in the mode coupling:

$$\mathbf{k}_i^2, \quad \mathbf{k}_i\cdot\mathbf{k}_j, \quad \mathbf{e}_z\cdot[\mathbf{k}_i\times\mathbf{k}_j], \qquad (3.5)$$

with $\mathbf{e}_z$ the unit vector in vertical direction. For systems which are invariant under reflections with respect to arbitrary planes perpendicular to the fluid layer, the dependence on the cross products drops out.

These symmetry considerations allow us to find suitable representations of the mode coupling coefficients. Let us first have a look at the quadratic mode coupling term Γ^2. This term depends on

$$\Gamma^2=\Gamma^2(\mathbf{k}^2,\mathbf{k}_1^2,\mathbf{k}_2^2,\mathbf{e}_z\cdot[\mathbf{k}_1\times\mathbf{k}_2]), \qquad (3.6)$$

since the scalar products as well as all other cross products can be expressed according to [Fig. 2(a)]):

$$\mathbf{k}_1\cdot\mathbf{k}_2=\tfrac{1}{2}((\mathbf{k}_1+\mathbf{k}_2)^2-\mathbf{k}_1^2-\mathbf{k}_2^2),$$
$$\mathbf{k}\cdot\mathbf{k}_1=\mathbf{k}_1^2+\mathbf{k}_1\cdot\mathbf{k}_2, \qquad (3.7)$$

$$\mathbf{k}\cdot\mathbf{k}_2=\mathbf{k}_2^2+\mathbf{k}_1\cdot\mathbf{k}_2,$$

$$\mathbf{e}_z\cdot[\mathbf{k}\times\mathbf{k}_1]=\mathbf{e}_z\cdot[\mathbf{k}_2\times\mathbf{k}_1],$$

$$\mathbf{e}_z\cdot[\mathbf{k}\times\mathbf{k}_2]=\mathbf{e}_z\cdot[\mathbf{k}_1\times\mathbf{k}_2].$$

Now let us turn to mode coupling terms of third order. For the following we shall assume that they arise solely due to elimination of stable modes, i.e., the nonlinearity of the system under consideration is quadratic. Then the mode coupling term is composed of a sum of products of the form [Fig. 2(b)]

$$A(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_3)B(\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3), \tag{3.8}$$

where each factor separatively is invariant with respect to translations and rotations. Applying the same reasoning as above we see that the coefficient A depends on

$$\mathbf{k}^2,\quad \mathbf{k_1}^2,\quad (\mathbf{k}_2+\mathbf{k}_3)^2,\quad \mathbf{e}_z\cdot(\mathbf{k}_1\times[\mathbf{k}_2+\mathbf{k}_3]), \tag{3.9}$$

whereas the coefficient B has the arguments

$$\mathbf{k}^2,\quad \mathbf{k_3}^2,\quad (\mathbf{k}_2+\mathbf{k}_3)^2,\quad \mathbf{e}_z\cdot(\mathbf{k}_2\times\mathbf{k}_3). \tag{3.10}$$

As a result the cubic mode coupling term takes the forms

$$\Gamma^3=\Gamma^3(\mathbf{k}^2,\mathbf{k}_1^2,\mathbf{k}_2^2,\mathbf{k}_3^2,(\mathbf{k}_2+\mathbf{k}_3)^2,\mathbf{e}_z\times(\mathbf{k}_1\times[\mathbf{k}_2+\mathbf{k}_3]),\mathbf{e}_z\cdot(\mathbf{k}_2\times\mathbf{k}_3)). \tag{3.11}$$

Up to now we have specified the dependences of the mode coupling coefficients Γ^2 and Γ^3 on the $\mathbf{k}$ vectors, taking into account the invariance of the system under translations, rotations, and reflections. This will allow us to perform approximations for the nonlinear mode coupling coefficients.

A. Isotropic and reflectionally invariant systems

Let us first consider a system which has, in addition to translational and rotational invariance, reflectional symmetry. Let us further assume that there are no quadratic nonlinearities. In that case the mode coupling term of third order takes the form

$$\Gamma^3=\Gamma^3(\mathbf{k}^2,\mathbf{k}_1^2,\mathbf{k}_2^2,\mathbf{k}_3^2,(\mathbf{k}_2+\mathbf{k}_3)^2). \tag{3.12}$$

First we restrict the patterns to be formed of N plane waves having wave vectors with exactly the critical value k_c,

$$A_n(t)=\xi(\mathbf{k}_n,t),\quad |\mathbf{k}_n|=k_c,\quad n=1,\dots,N. \tag{3.13}$$

The following system of Landau type amplitude equations are derived from the evolution equation (2.9):

$$\dot{A}_n(t)=\lambda(k_c^2)A_n(t)-\sum_{m=1}^{N} b(\Theta_{mn})|A_m(t)|^2A_n(t). \tag{3.14}$$

Here the mode coupling coefficient b as well as the angle Θ_{mn} between two wave vectors $\mathbf{k}_m$ and $\mathbf{k}_n$ have been introduced [fig. 2(b)]. It is related to the coefficient Γ^3 in the following way:

$$b(\theta_{mn})=-\Gamma^3(k_c^2;k_c^2,k_c^2,k_c^2,2k_c^2[\cos(\Theta_{mn})-1]). \tag{3.15}$$

We shall assume that the Taylor expansion of $b(\theta_{mn})$ in $\cos(\theta_{mn})-1$ converges. This assumption allows us to perform a formal Taylor expansion of the nonlinear mode coupling coefficient $\Gamma^3$with respect to the variable $(\mathbf{k}_2+\mathbf{k}_3)^2$ at the origin $(\mathbf{k}_2+\mathbf{k}_3)^2=0$. Furthermore, we take into account that the dominant contributions to the evolving pattern stem from plane waves with absolute values of the $\mathbf{k}$ vectors close to the critical one. Therefore, we may release the assumption $|\mathbf{k}_i|=k_c$ and expand Γ^3 with respect to $\mathbf{k}_i^2$ at k_c^2. Our formal result reads

$$\Gamma^3=\sum_{n;ijlm} B_{n;ijlm}(-1)^n(k_c^2-\mathbf{k}^2)^i(k_c^2-\mathbf{k}_1^2)^j \times(k_c^2-\mathbf{k}_2^2)^l(k_c^2-\mathbf{k}_3^2)^m(\mathbf{k}_2+\mathbf{k}_3)^{2n}. \tag{3.16}$$

Now we may transform to real space according to Eq. (2.11). The order parameter equation for the field $\Psi(\mathbf{x},t)$ then reads, with the linear part given in (3.2),

$$\dot{\Psi}(\mathbf{x},t)=[\varepsilon-\tilde{\Delta}^2]\Psi(\mathbf{x},t)+\sum_{n;ijlm} B_{n;ijlm}\tilde{\Delta}^i\{\tilde{\Delta}^j\Psi(\mathbf{x},t)\Delta^n[\tilde{\Delta}^l\Psi(\mathbf{x},t)\tilde{\Delta}^m\Psi(\mathbf{x},t)]\}, \tag{3.17}$$

with the abbreviation

$$\tilde{\Delta}=1+\Delta. \tag{3.18}$$

Note that the spatial coordinates are scaled so that the critical wave length is 2π, i.e., $k_c=1$. A truncation of the formal Taylor expansion leads to an evolution equation with local nonlinear interaction terms, i.e., terms involving only the field $\Psi(\mathbf{x},t)$ as well as its spatial derivatives. The simplest approximation using only the term $B_{0;000}$ leads to the Swift-Hohenberg equation [25]. In the appendix we shall consider roll solutions of Eq. (3.17), and discuss their stability with respect to long wavelength disturbances as well as with respect to amplitude instabilities.

It is interesting to notice that the order parameter equation turns into a gradient system if the coefficients $B_{n;ijlm}$ are totally symmetric with respect to the indices $i,j,l,$ and m. In that case the following Ljapunov potential exists:

$$V^4[\Psi]=-\tfrac{1}{2}\int d\mathbf{x}[(\varepsilon-\tilde{\Delta})\Psi(\mathbf{x},t)]^2-\tfrac{1}{4}\sum_{n;ijlm} B_{n;ijlm}\int d\mathbf{x}[\tilde{\Delta}^i\Psi(\mathbf{x},t)][\tilde{\Delta}^j\Psi(\mathbf{x},t)]\Delta^n\{[\tilde{\Delta}^l\Psi(\mathbf{x},t)][\tilde{\Delta}^m\Psi(\mathbf{x},t)]\}, \tag{3.19}$$

from which the evolution equation (3.17) is obtained by the functional derivative

$$\dot{\Psi}=-\frac{\delta V^4[\Psi]}{\delta\Psi}. \tag{3.20}$$

Furthermore, if one is content with the lowest order approximation $\varepsilon^{(3/2)}$, neglecting contributions of terms of the order of

$$\tilde{\Delta}^i\Psi(\mathbf{x},t), \quad i>0 \tag{3.21}$$

to the cubic part, the terms $B_{n;0000}$ are totally symmetric and a Ljapunov potential having the form

$$V^4[\Psi]=-\tfrac{1}{2}\int d\mathbf{x}[(\varepsilon-\tilde{\Delta})\Psi(\mathbf{x},t)]^2 -\tfrac{1}{4}\sum_{n} B_{n;0000}\int d\mathbf{x}\,\Psi^2(\mathbf{x},t)\Delta^n\Psi^2(\mathbf{x},t) \tag{3.22}$$

exists.

B. Isotropic systems lacking reflectional symmetry

Now we release the assumption of reflectional symmetry. As an example we mention the convective instability in large-aspect-ratio systems rotating around a vertical axis [27–29]. This externally applied rotation conserves rotational invariance but breaks reflection symmetry.

Again we perform a Taylor expansion with respect to the variables $\mathbf{k}_i^2$ at k_c^2 and $(\mathbf{k}_2+\mathbf{k}_3)^2$, $\mathbf{e}_3\cdot[\mathbf{k}_2\times\mathbf{k}_3]$, and $\mathbf{e}_3\cdot[\mathbf{k}_1\times(\mathbf{k}_2+\mathbf{k}_3)]$ at zero. We introduce the abbreviation

$$\alpha_i=k_c^2-\mathbf{k}_i^2. \tag{3.23}$$

Furthermore, we note that even powers of terms involving the cross products are invariants with respect to reflections, and can therefore be expressed in terms of powers of $\mathbf{k}_1^2$, $\mathbf{k}_2^2$, and $(\mathbf{k}_1+\mathbf{k}_2)^2$. Therefore, the expansion reads

$$\Gamma^3=\sum_{n;ijlm} B_{n;ijlm}(-1)^n\alpha^i\alpha_1^j\alpha_2^l\alpha_3^m(\mathbf{k}_2+\mathbf{k}_3)^{2n}+\sum_{n;ijlm} C_{n;ijlm}(-1)^n\alpha^i\alpha_1^j\alpha_2^l\alpha_3^m(\mathbf{k}_2+\mathbf{k}_3)^{2n}\mathbf{e}_z\cdot(\mathbf{k}_2\times\mathbf{k}_3)$$
$$+\sum_{n;ijlm} D_{n;ijlm}(-1)^n\alpha^i\alpha_1^j\alpha_2^l\alpha_3^m(\mathbf{k}_2+\mathbf{k}_3)^{2n}\mathbf{e}_z\cdot(\mathbf{k}_1\times[\mathbf{k}_2+\mathbf{k}_3]). \tag{3.24}$$

Transforming back to real space yields the desired order parameter equation

$$\dot{\Psi}(\mathbf{x},t)=[\varepsilon-\tilde{\Delta}^2]\Psi(\mathbf{x},t)+\sum_{n;ijlm} B_{n;ijlm}\tilde{\Delta}^i\{\tilde{\Delta}^j\Psi(\mathbf{x},t)\Delta^n[\tilde{\Delta}^l\Psi(\mathbf{x},t)\tilde{\Delta}^m\Psi(\mathbf{x},t)]\}$$
$$+\sum_{n;ijlm} C_{n;ijlm}\tilde{\Delta}^i\{\tilde{\Delta}^j\Psi(\mathbf{x},t)\Delta^n\mathbf{e}_z\cdot[\nabla\tilde{\Delta}^l\Psi(\mathbf{x},t)\times\nabla\tilde{\Delta}^m\Psi(\mathbf{x},t)]\}$$
$$+\sum_{n;ijlm} D_{n;ijlm}\tilde{\Delta}^i\{\mathbf{e}_z\cdot[\nabla\tilde{\Delta}^j\Psi(\mathbf{x},t)\times\nabla\Delta^n[\tilde{\Delta}^l\Psi(\mathbf{x},t)\tilde{\Delta}^m\Psi(\mathbf{x},t)]]\}. \tag{3.25}$$

We mention that equations of the present type have been applied to describe pattern formation in rotating Bénard convection where complex spatio-temporal patterns arise due to the Küppers-Lortz instability [30–32].

C. Quadratic terms

Now we wish to include also quadratic terms into the evolution equation (2.9). Let us again first consider the case of a reflectionally invariant system. Then the dependence on the cross product $\mathbf{e}_z\cdot[\mathbf{k}_2\times\mathbf{k}_3]$ drops out, and we can perform an expansion in $\mathbf{k}_i^2$ at k_c^2.

$$\Gamma^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)=\sum_{ijl} C_{ijl}\alpha^i\alpha_1^j\alpha_2^l. \tag{3.26}$$

For systems lacking reflectional symmetry, we have to take into account a term proportional to the cross product $\mathbf{e}_z\cdot[\mathbf{k}_2\times\mathbf{k}_3]$:

$$\Gamma^2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)=\sum_{ijl} D_{ijl}\alpha^i\alpha_1^j\alpha_2^l\mathbf{e}_z\cdot[\mathbf{k}_1\times\mathbf{k}_2]. \tag{3.27}$$

Transforming back to real space yields the two quadratic terms

$$\sum_{ijl} C_{ijl}\tilde{\Delta}^i[\tilde{\Delta}^j\Psi(\mathbf{x},t)\tilde{\Delta}^l\Psi(\mathbf{x},t)],$$

$$\sum_{ijl} D_{ijl}\tilde{\Delta}^i\mathbf{e}_z\cdot[\boldsymbol{\nabla}\tilde{\Delta}^j\Psi(\mathbf{x},t)\times\boldsymbol{\nabla}\tilde{\Delta}^l\Psi(\mathbf{x},t)]. \tag{3.28}$$

It has been demonstrated that the inclusion of quadratic terms leads to the formation of hexagonal patterns close to onset of the instability [33,34].

D. Order parameters of pseudoscalar type

It is well established that the convective instability in systems with low Prandtl numbers is not governed only by an order parameter which belongs to the convective roll structure. As first noticed by Siggia and Zippelius [35], there are effects of large scale horizontal drift motions due to the fact that large scale vortical motions are only weakly damped. A secondary spatially slowly varying field arises which has to be considered as a further order parameter. This situation has been modeled by introducing the stream function of the vortical velocity field as a second order parameter field. However, the stream function is pseudoscalar. Therefore, we shall discuss a pattern forming system for which an order parameter of pseudoscalar type has to be taken into account. This observation will lead us to generalizations of the above form of the model of Manneville [36,37].

In addition to an order parameter field arising due to an instability at a finite wave vector k_c, we consider a secondary order parameter field $\Phi(\mathbf{x},t)$ which is pseudoscalar. Furthermore, we assume that the two fields are coupled by quadratic terms. For this case, the set of order parameter equations reads

$$\dot{\Psi}(\mathbf{x},t)=\lambda(\tilde{\Delta})\Psi(\mathbf{x},t)+\sum_{n;ijlm}B_{n;ijlm}\tilde{\Delta}^i\{\tilde{\Delta}^j\Psi(\mathbf{x},t)\Delta^n[\tilde{\Delta}^l\Psi(\mathbf{x},t)\tilde{\Delta}^m\Psi(\mathbf{x},t)]\}+\sum_{i;jl}C_{i;jl}\tilde{\Delta}^i\mathbf{e}_3\cdot[\boldsymbol{\nabla}\tilde{\Delta}^j\Psi(\mathbf{x},t)\times\boldsymbol{\nabla}\Delta^l\Phi(\mathbf{x},t)], \tag{3.29}$$

$$\tau(\Delta)\dot{\Phi}(\mathbf{x},t)=\gamma(\Delta)\Phi(\mathbf{x},t)+\sum_{n;ijlm}D_{i;jl}\tilde{\Delta}^i\mathbf{e}_3\cdot[\boldsymbol{\nabla}\tilde{\Delta}^j\Psi(\mathbf{x},t)\times\boldsymbol{\nabla}\tilde{\Delta}^l\Psi(\mathbf{x},t)]. \tag{3.30}$$

Here $\gamma(-k^2)/\tau(-k^2)$ denotes the linear growth rates of the normal modes of the pseudoscalar field $\Phi(\mathbf{x},t)$. We mention that quadratic terms involving the fields Φ and Ψ as well as Φ and Φ could also arise in the equation for Φ. The structure of the corresponding coupling coefficients can be obtained by similar reasonings, so that we do not need to specify them here.

The class of systems described by the present types of order parameter equations have become highly important by the observation that they are able to describe the so-called spiral turbulence [38], which has been investigated experimentally [39], and recently in direct numerical simulations of the three dimensional hydrodynamic equations [40].

IV. CONCLUSION

In the present paper we have given a detailed derivation of order parameter equations describing the evolution of patterns in nonequilibrium systems. Starting from a general set of equations of motion, we argued that the reduction of the many degrees of freedom to a few ''relevant'' ones, namely, the order parameters, close to an instability leads to a nonlocal order parameter equation in form of an integrodifferential equation. We have shown that the nonlocal terms may be approximated by a suitable local gradient expansion. This expansion takes into account that close to threshold the order parameter field is excited only in a finite band around a typical critical wave number. This allows us to perform a systematic expansion of the nonlocal terms in powers of the bandwidth. A secondary expansion takes into account the nonlinear interaction between plane waves of different directions.

APPENDIX: STABILITY BOUNDARIES OF ROLLS

Here we want to investigate the stability of roll solutions for the order parameter equation (3.17). In the lowest order of ε, a solution of Eq. (3.17) having the form of parallel rolls reads

$$\Psi_0(\mathbf{x})=A_0(k)\sin(kx), \tag{A1}$$

with

$$A_0^2(k)=-4\frac{\varepsilon-\alpha^2}{c_0(k)}, \tag{A2}$$

where α is given in Eq. (3.23), and the abbreviation

$$c_0(k)=\sum_{ijlmn}B_{n;ijlm}\alpha^{i+j+l+m}[(-4k^2)^n+2\delta_{n0}] \tag{A3}$$

with the Kronecker symbol δ_{ij} is used.

1. Phase instabilities

To obtain the phase instability boundaries that confine the regions in the k-ε plane, where Eq. (A1) is stable with respect to disturbances with long wavelength we invoke the method of phase equations [41,4]. We insert

$$\Psi(\mathbf{x},t)=\left(\frac{A_0(k)}{2}+a(x,t)\right)e^{i(kx+\Phi(\mathbf{x},t))}, \tag{A4}$$

with the spatially slowly varying amplitude $a(x,t)$ and phase $\Phi(\mathbf{x},t)$ into Eq. (3.17). After linearization and neglecting higher spatial derivatives we obtain equations of the forms

$$\dot{a}=f(a,\partial_x\Phi),$$
$$\dot{\Phi}=g(\partial_{xx}\Phi,\partial_{yy}\Phi,a,\partial_x a), \tag{A5}$$

where f and g are linear functions of Φ and a and their derivatives. The assumption that the amplitude is enslaved by the phase allows an adiabatic elimination of the first by solving

$$f(a,\partial_x\Phi)=0 \tag{A6}$$

for a, which yields the instantaneous relation $a=a(\partial_x\Phi)$. Inserting this into the equation of motion for the phase (A5) leads to the phase equation

$$\dot{\Phi}(\mathbf{x},t)=[D_\parallel\partial_{xx}+D_\perp\partial_{yy}]\Phi(\mathbf{x},t). \tag{A7}$$

The phase diffusion coefficients can be computed explicitly from Eq. (3.17). They read

$$\begin{aligned}D_\parallel={}&4k^2\alpha\left(\frac{c_1(k)}{c_0(k)}-\frac{2\alpha}{\varepsilon-\alpha^2}\right)+6k^2-2-\frac{1}{c_0(k)}\sum_{ijlmn}B_{n;ijlm}\alpha^{i+j+l+m-2}\\&\times\Bigg\{\Bigg[\frac{1}{2}(-4k^2)^{n+1}(i(i-1)-j(j-1)+l(l-1)+m(m-1)+2i(j+l+m))-\alpha(-4k^2)^n(j-l-m-i-2n(2i+l+m))\\&+2\alpha^2(-4k^2)^{n-1}n(2n-1)-4k^2\delta_{n0}(i(i-1)+j(j-1)+2i(j+l+m))+2\alpha\delta_{n0}(i+j)\Bigg][\varepsilon-\alpha^2]-2\alpha k^2[2\delta_{n0}(3i+j)\\&+(-4k^2)^n(3i-j+l+m)+4n\alpha(-4k^2)^{n-1}]\left[\frac{c_1(k)}{c_0(k)}(\varepsilon-\alpha^2)-2\alpha\right]\Bigg\}\end{aligned} \tag{A8}$$

and

$$D_\perp=-2\alpha-\frac{\varepsilon-\alpha^2}{c_0(k)}\sum_{ijlmn}B_{n;ijlm}\alpha^{i+j+l+m-1}[(-4k^2)^n(i-j+l+m)+2n\alpha(-4k^2)^{n-1}+2\delta_{n0}(i+j)], \tag{A9}$$

with

$$c_1(k)=\frac{dc_0(k)}{dk^2}.$$

Rolls are unstable if at least one diffusion coefficient becomes negative. Therefore,

$$D_\parallel=0 \tag{A10}$$

denotes the longitudinal or *Eckhaus* instability [42,43], and

$$D_\perp=0 \tag{A11}$$

the vertical or zigzag instability [43].

Assuming again that the width of the excited band of the order parameter in Fourier space depends on ε, i.e.,

$$\alpha\propto\varepsilon^{1/2}, \tag{A12}$$

simplifies expressions (A11) and (A10). If we solve for ε we obtain up to the appropriate order the stability boundaries

$$\varepsilon_\parallel=\left(1+\frac{4k^2}{3k^2-1}\right)\alpha^2+O(\alpha^3),\quad \varepsilon_\perp=-2\frac{a}{b}\alpha+\left(\frac{2ab_1}{b^2}-\frac{2a_1}{b}\right)\alpha^2+O(\alpha^3), \tag{A13}$$

(A13)

where we used the abbreviations

$$a \equiv \sum_n B_{n;0000}((-4k^2)^n + 2\delta_{n0}), \tag{A14}$$

$$a_1 \equiv \sum_{ijlmn} B_{n;ijlm} \delta_{i+j+l+m,1}((-4k^2)^n + 2\delta_{n0}),$$

$$b \equiv \sum_{ijlmn} B_{n;ijlm}[(-4k^2)^n(i-j+l+m) + 2\delta_{n0}(i+j)]\delta_{i+j+l+m,1},$$

$$b_1 \equiv \sum_{ijlmn} B_{n;ijlm}[(-4k^2)^n(i-j+l+m) + 2\delta_{n0}(i+j)][\delta_{i+j+l+m,2} + 2n(-4k^2)^{n-1}\delta_{i+j+l+m,1}].$$

From Eq. (A13) it is obvious that in lowest self consistent order the longitudinal instability boundary cannot depend on the nonlinear coefficients of the gradient expansion (3.17). The Eckhaus instability near threshold is universal and independent from the special form of the kernel Γ^3. A quite different situation occurs for the zigzag instability. Here all coefficients with respect to the index n as well as the two lowest order fulfilling the condition $i+j+l+m \leqslant 2$ are important, and may influence the shape of the boundary.

2. Amplitude instabilities

In contrast to the long wavelength phase instabilities, the amplitude instabilities usually have a wave vector comparable or close to the critical one, but a different direction. Again we examine the stability of parallel rolls [Eq. (A1)], now making the ansatz

$$\Psi(\mathbf{x},t) = \Psi_0(\mathbf{x}) + a e^{(i\mathbf{k}_1\mathbf{x} + \sigma t)}, \tag{A15}$$

with arbitrary orientated $\mathbf{k}_1$ for Eq. (3.17). Linearization with respect to a gives the growth rate

$$\sigma = \varepsilon - \alpha_1^2 - (\varepsilon - \alpha^2)\frac{c(\mathbf{k},\mathbf{k}')}{c_0(k)}, \tag{A16}$$

with

$$\alpha_1 = 1 - k_1^2, \quad \mathbf{k} = (k,0) \tag{A17}$$

and

$$c(\mathbf{k},\mathbf{k}') = \sum_{ijlmn} B_{n;ijlm}\{(\alpha_1^{i+l}\alpha^{j+m} + \alpha_1^{i+m}\alpha^{j+l}) \times ([-(\mathbf{k}+\mathbf{k}')^2]^n + [-(\mathbf{k}-\mathbf{k}')^2]^n) + 2\delta_{n0}\alpha_1^{i+j}\alpha^{l+m}\}. \tag{A18}$$

The condition $\sigma = 0$ yields the stability boundaries with respect to the amplitude instability. To obtain the most dangerous mode, the length and direction of $\mathbf{k}_1$ must be chosen so that σ has a maximum. The terms in the sum over n in Eq. (A18) with $n=0$ and 1 have no angular dependence and waves with arbitrary directions become unstable simultaneously. This is also the case for the Swift-Hohenberg equation $(n=0)$. The term for $n=2$ is the first one that leads to an angular dependent growth rate (A16) and must be included to adjust the stability boundary. It is easily shown that this expression can produce a maximal growth rate at an angle of 90° between $\mathbf{k}$ and $\mathbf{k}_1$, leading to an instability that sets in perpendicularly to Ψ_0, which is nothing else than the well-known cross-roll (CR) instability [43].

Again we may expand σ in powers of ε [or, according to Eq. (A12), of α^2]. Up to order ε we obtain

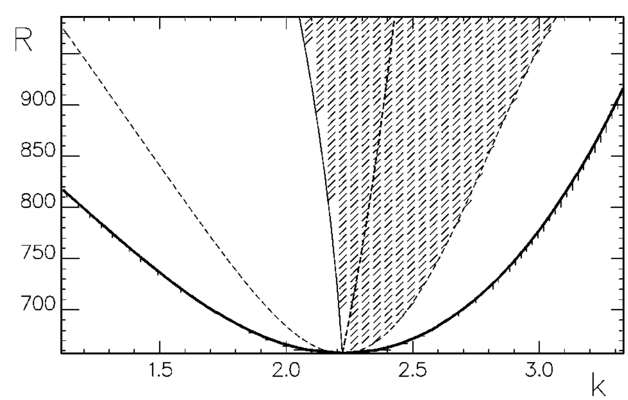

FIG. 3. Stability range of convection rolls in the Rayleigh-Bénard problem calculated from Eq. (3.17). The coefficients of Eq. (3.17) were computed for the case of free vertical boundary conditions and infinite Prandtl number. The Rayleigh number is denoted by R. Convection sets in above the bold line. Rolls with a wave vector $\mathbf{k}$ are stable in the shaded area and are bounded from the left hand side by the zigzag instability (solid), and from the right hand side by the cross-roll instability (dashed). The bold dashed line denotes k_c, which varies slightly with R.

$$\sigma_{\mathrm{CR}}=\varepsilon-\alpha'^2-2(\varepsilon-\alpha^2)\times\frac{\sum_n B_{n;0000}([-(\mathbf{k}+\mathbf{k}')^2]^n+[-(\mathbf{k}-\mathbf{k}')^2]^n+\delta_{n0})}{\Sigma_n B_{n;0000}((-4k^2)^n+2\delta_{n0})}. \tag{A19}$$

The boundary of the cross-roll instability is influenced only by the coefficients belonging to the expansion with respect to Δ^n in Eq. (3.17).

In summary we see that if the expansion with respect to the index n converges rapidly, model equations obtained by low order truncations of the nonlinear interaction terms are reasonable approximations in the sense that they yield converging expressions for the stability boundaries of the roll solutions.

3. Busse balloon

As an example we computed the phase and amplitude instabilities of parallel rolls as the first instability in Rayleigh-Bénard convection from Eq. (3.17). First order terms with respect to the bandwidth $\tilde{\Delta}$ as well as second order terms of the angular coupling Δ^n have been taken into account. For the case of free boundaries in vertical direction the linear problem (2.3) may be solved analytically. The coefficients $B_{n;ijlm}$ of Eq. (3.17) are obtained from Eq. (3.12), which takes the following form:

$$\Gamma^3(k^2,k_1^2,k_2^2,k_3^2,k_s^2)=-\frac{R_c^2}{256\lambda^{(2)}(k_s^2)\pi^2(\pi^2+k_2^2)^2}\times\left[\frac{k_s^2-k^2-3k_1^2}{(\pi^2+k_1^2)^2}+2\frac{k_3^2-k_2^2}{(4\pi^2+k_s^2)^2}\right]\times[3k_2^2+k_3^2-k_s^2], \tag{A20}$$

with $\mathbf{k_s}\equiv\mathbf{k}_2+\mathbf{k}_3$ and $\lambda^{(2)}$ being the eigenvalue of the first stable mode having the z dependence $\sin(2\pi z)$.

Figure 3 shows the results. Note that compared to the diagrams found from the Swift-Hohenberg equation [25] the zigzag instability is inclined to the left, and cross-roll and Eckhaus instabilities are changed. Rolls are bounded for large wave vectors by the cross-roll instability. The diagram is in excellent agreement with that computed directly from the hydrodynamic equations [44].

[1] H. Haken, *Synergetics. An Introduction*, 3rd ed. (Springer, Berlin, 1983).
[2] A. C. Newell, T. Passot, and J. Lega, Annu. Rev. Fluid Mech. **25**, 399 (1993).
[3] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[4] P. Manneville, *Dissipative Structures and Weak Turbulence* (Academic, San Diego, 1990).
[5] *Nonlinear Evolution of Spatiotemporal Strutures in Dissipative Continuous Systems*, edited by F. H. Busse and L. Kramer (Plenum, New York, 1990).
[6] F. H. Busse, Rep. Prog. Phys. **41**, 1929 (1978).
[7] L. Kramer and W. Pesch, Annu. Rev. Fluid Mech. **27**, 515 (1995).
[8] R. Friedrich, M. Bestehorn, and H. Haken, Int. J. Mod. Phys. B **4**, 365 (1990).
[9] W. S. Edwards and S. Fauve, Phys. Rev. E **47**, R788 (1993).
[10] A. M. Turing, Philos. Trans. R. Soc. London, Ser. B **327**, 37 (1952).
[11] V. Castets, E. Dulos, J. Boissonade, and P. De Kepper, Phys. Rev. Lett. **64**, 2953 (1990).
[12] Q. Ouyang and H. L. Swinney, Nature (London) **352**, 610 (1991).
[13] F. T. Arecchi, Physica D **51**, 450 (1991).
[14] H. Haken, *Advanced Synergetics*, 2nd ed. (Springer, Berlin, 1987).
[15] D. H. Sattinger, *Group Theoretic Methods in Bifurcation Theory*, Lecture Notes in Mathematics Vol. 762 (Springer, New York, 1980).
[16] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory* (Springer, New York, 1988).
[17] A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).
[18] A. C. Newell, in *Propagation in Systems Far from Equilibrium*, edited by J. E. Wesfreid, H. R. Brand, P. Manneville, G. Albinet, and N. Boccara, Springer Series in Synergetics Vol. 41 (Springer, New York, 1988), p. 122.
[19] L. A. Segel, J. Fluid Mech. **19**, 203 (1969).
[20] A. Nepomnyashchy, Int. J. Bifurcation Chaos Appl. Sci. Eng. **4**, 1147 (1994).
[21] H. Haken, Rev. Mod. Phys. **47**, 67 (1975).
[22] P. Coullet, K. Emilsson, and D. Walgraef, Physica D **61**, 132 (1992).
[23] B. Malomed, A. A. Nepomnyashchy, and M. I. Tribelsky, Phys. Rev. A **42**, 7244 (1990).
[24] P. Collet and J.-P.Eckmann, *Instabilities and Fronts in Extended Systems*, Princeton Series in Physics Vol. VII (Princeton University Press, Princeton, 1990).
[25] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).
[26] M. Bestehorn, R. Friedrich, and H. Haken, Z. Phys. B **75**, 265 (1989).
[27] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon, Oxford, 1961).
[28] G. Küppers and D. Lortz, J. Fluid Mech. **35**, 609 (1969).
[29] F. H. Busse and K. E. Heikes, Science **208**, 173 (1980).
[30] M. Fantz, R. Friedrich, M. Bestehorn, and H. Haken, in New Trends in Nonlinear Dynamics: Nonvariational Aspects, edited by C. Pérez-García [Physica D **61**, 147 (1992)].
[31] M. Neufeld, R. Friedrich, and H. Haken, Z. Phys. B **92**, 243 (1993).
[32] M. Neufeld and R. Friedrich, Phys. Rev. E **51**, 2033 (1995).
[33] M. Bestehorn and H. Haken, Phys. Lett. A **99**, 265 (1983).
[34] M. Bestehorn, Phys. Rev. E **48**, 3622 (1993).
[35] E. D. Siggia and A. Zippelius, Phys. Rev. Lett. **47**, 835 (1981).
[36] P. Manneville, J. Phys. (Paris) **44**, 759 (1983).

[37] P. Manneville, J. Phys. (Paris) **44**, L-903 (1983).
[38] M. Bestehorn, M. Fantz, R. Friedrich, and H. Haken, Phys. Lett. A **174**, 48 (1993).
[39] S. W. Morris, E. Bodenschatz, D. S. Cannell, and G. Ahlers, Phys. Rev. Lett. **71**, 2026 (1993).
[40] W. Pesch, Chaos **6**, 348 (1996).
[41] Y. Pomeau and P. Manneville, J. Phys. (Paris) **42**, 1067 (1981).
[42] W. Eckhaus, *Studies in Nonlinear Stability Theory* (Springer, New York, 1985).
[43] F. H. Busse, J. Math. Phys. **46**, 140 (1967).
[44] F. H. Busse and E. W. Bolton, J. Fluid Mech. **146**, 115 (1984).